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Self-energy of the graviton in second order

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Abstract. Following a recently developed approach to the problem of divergences in field theory, we compute the second-order correction to the graviton Green function arising from insertion of the graviton loop. As was the case with the calculations reported earlier, *no subtractions, regularizations or renormalizations are required. Our corrected graviton* Green function depends only logarithmically on a constant l and satisfies the appropriate Slavnov-Ward identity (SWI). The absence of divergences means that we do not need to introduce counter terms into the Einstein Lagrangian. It also allows a straightforward and expeditious evaluation of the self-energy insertion. In addition, we report the calculation of the correction to the graviton Green function arising from the insertion of a scalar meson loop. Again there are no divergences. However, in this case we find that the result violates the SWI. On the other hand, if the gravitons are coupled to the scalar field through the improved energy tensor of Callan, Coleman and Jackiw, the corrected graviton Green function does satisfy the SWI. The correction as well as the self-energy is traceless. Thus there is no conformal anomaly.

1. Introduction

Since the formulation of the quantum theory of gravitation by Feynman (1963) and DeWitt (1967), the general framework for evaluating radiative corrections to processes involving gravitons has been available. However, the divergences associated with loop graphs in quantum gravity are more severe than in other theories of physical interest. Thus, it is correspondingly more difficult to isolate the finite parts of the amplitudes in a manner consistent with the symmetries of the theory. Actual calculations of radiative corrections had to await the development of the dimensional regularization (DR) method by 't Hooft and Veltman (1972). This method has the great merit of maintaining gauge invariance throughout the calculations. Hence, it is much better suited for Yang-Mills fields and quantum gravity and is almost exclusively used in all works. However, the age-old unsolved problem of divergences must still be dealt with before one can make theoretical predictions. In the DR method, the pole terms must be cancelled by suitably chosen counter terms in the Lagrangian which likewise diverge as the *dimensions of spacetime approach four. In case of quantum gravity the required* terms have a different dependence on the field variables from those present in the original Lagrangian. Consequently, the divergences can not be absorbed by field renormalization. Deser and van Nieuwenhuizen (1974) have shown the Einstein-Maxwell fields and Dirac-Einstein fields to be non-renormalizable at the one-loop level. Capper *et al* (1974) and Capper and Duff (1973) computed respectively the photon and neutrino corrections to the graviton propagator using the DR method and found non-renormalizable counterterms. However, 't Hooft and Veltman (1974) have shown pure gravity to be renormalizable at the one-loop level.

In order to understand how nature avoids divergence problems which seem inevitable in our current formulation of field theory, it is necessary to introduce new assumptions and modify the formalism. Perhaps it is necessary to question some of our most cherished notions about the way physical influences propagate in spacetime. This is a problem with a long history. Therefore, it is conceivable that to make progress one must at first tolerate incomplete solutions whose only purpose in hindsight might be to point the way. At present, there does not seem to be any empirical evidence to suggest a physical mechanism that eliminates divergences. Accordingly, such assumptions are necessarily *ad hoc*. In fact, the regularization methods represent *ad hoc* assumptions that are not necessarily specified in physical terms and are justified only by their limited success in dealing with divergences. We emphasize that taking the limit of infinite regulator masses or the limit of spacetime dimension equal to four as the case may be, does not ameliorate the qualitative modification of the field theory caused by the regularization procedure.

The method proposed recently (Zaidi 1990) makes the explicit physical assumption that the product of two scalar propagators with equal arguments abruptly vanishes if their argument, in Euclidean space, is less than a natural constant $4l^2$. Clearly, this assumption cannot be confirmed or refuted by experiment if the natural constant is small enough. Six two-point functions from theories of physical interest were computed in second order using this assumption. These included, the Green function of a vector meson in a typical Yang–Mills theory, and the photon and neutrino loop corrections to the graviton Green function. All these results depended logarithmically on the constant l, satisfied the appropriate SWI and had the imaginary part required by unitarity for *finite* values of l. The results generally agreed with those obtained by the DR method with the exception of the pole terms that are absent in our approach. In addition, the calculation of the photon-loop and neutrino-loop insertions based on DR method leads to conformal anomalies whereas our corrections to the graviton Green function and the self-energies were traceless. As our results were free of divergences, no regularizations or renormalizations were required. Inasmuch as the quantity l occurred only in the arguments of logarithms and determined the relationships between the physical values of coupling constants and masses, and the parameters of the Lagrangian it is entirely feasible to take it to be as small as the Planck length.

In this paper, we report the calculations of the corrections to the graviton Green function arising from the graviton-loop and the scalar-loop respectively. The graviton-loop calculation is naturally somewhat lengthy but the absence of singular expressions in our approach makes it a straightforward exercise.

2. Graviton self-energy insertion

The graviton self-energy was calculated by Capper *et al* (1973) using the DR method. They gave a detailed formulation of quantum gravity based on the Einstein Lagrangian and expanded the action in powers of Newton's constant. They also derived the SWI. We briefly summarize their development and adopt their notation as far as possible. The generating functional is given by

$$Z[j_{\mu\nu}] = \int \mathrm{D}\tilde{g}^{\mu\nu}\, \Delta[\tilde{g}^{\mu\nu}] \exp(\mathrm{i}\langle -2K^{-2}\tilde{g}^{\mu\nu}R_{\mu\nu} + K^{-1}g^{\mu\nu}j_{\mu\nu} - K^{-2}\alpha^{-1}(\partial_\mu \tilde{g}^{\mu\nu})^2\rangle). \qquad (1)$$

Here $K^2 = 32\pi G$ and G is Newton's constant. The angle brackets denote integration over Minkowski space. In addition, the scalar curvature has been expressed in terms of the variable $\tilde{g}^{\mu\nu}$ introduced by Goldberg (1958) which is defined as

$$\tilde{g}^{\mu\nu}=\sqrt{g}\, g^{\mu\nu} \tag{(2)}$$

and $g\equiv-\det(g_{\mu\nu})$. The Fadeev–Popov determinant is denoted by $\Delta[\tilde{g}^{\mu\nu}]$ and the gauge fixing term is $K^{2-}\alpha^{-1}(\partial_\mu\tilde{g}^{\mu\nu})^2$, where α is the gauge parameter.

Defining the graviton field $\phi^{\mu\nu}$ by the equation

$$\tilde{g}^{\mu\nu}=\eta^{\mu\nu}+K\phi^{\mu\nu} \tag{3}$$

and expanding the action in powers of K one can write

$$I_G=I_G^{(0)}+I_G^{(1)}. \tag{4}$$

The flat space metric tensor corresponds to the choice $\mathrm{Tr}(\eta_{\mu\nu})=2$. From this point onwards, indices are raised and lowered using $\eta_{\mu\nu}$. However, in order to simplify the notation, all the indices are written as subscripts. Accordingly, before summing over repeated indices one should imagine one of the dummy indices to be a superscript. Thus, we have the expressions

$$I_G^{(0)}=\langle-\tfrac{1}{2}\phi_{\alpha\beta,\sigma}\phi_{\alpha\beta,\sigma}+\tfrac{1}{4}\phi_{\alpha\alpha,\sigma}\phi_{\beta\beta,\sigma}+\phi_{\alpha\kappa,\rho}\phi_{\alpha\rho,\kappa}-\phi_{\alpha\beta,\alpha}\phi_{\sigma\beta,\sigma}\rangle \tag{5}$$

and

$$I_G^{(1)}=\frac{K}{2}\langle\phi_{\sigma\rho}(\phi_{\alpha\beta,\sigma}\phi_{\alpha\beta,\rho}-\tfrac{1}{2}\phi_{\alpha\alpha,\sigma}\phi_{\beta\beta,\rho}+2\phi_{\sigma\alpha,\beta}\phi_{\rho\beta,\alpha}$$
$$-2\phi_{\sigma\alpha,\beta}\phi_{\rho\alpha,\beta}+\phi_{\sigma\rho,\alpha}\phi_{\beta\beta,\alpha})\rangle. \tag{6}$$

Next, $I_G^{(0)}$ is cast in the form

$$I_G^{(0)}=\tfrac{1}{2}\langle\phi_{\alpha\beta}O_{\alpha\beta\lambda\mu}\phi_{\lambda\mu}\rangle, \tag{7}$$

where

$$O_{\alpha\beta\lambda\mu}=\tfrac{1}{2}(\eta_{\alpha\lambda}\eta_{\beta\mu}+\eta_{\alpha\mu}\eta_{\beta\lambda}-\eta_{\alpha\beta}\eta_{\lambda\mu})\partial^2. \tag{8}$$

In writing (5) and (8) the gauge parameter α has been set equal to -1. The expression for the zeroth-order generating functional becomes

$$Z^{(0)}[j_{\mu\nu}]=\exp\left(-\frac{\mathrm{i}}{2}\langle j_{\alpha\beta}D^{(0)}_{\alpha\beta\mu\nu}j_{\mu\nu}\rangle\right). \tag{9}$$

Here

$$D^{(0)}_{\alpha\beta\mu\nu}=b_{\alpha\beta\mu\nu}D \tag{10}$$

$$b_{\alpha\beta\mu\nu}=\tfrac{1}{2}(\eta_{\alpha\mu}\eta_{\beta\nu}+\eta_{\alpha\nu}\eta_{\beta\mu}-\eta_{\alpha\beta}\eta_{\mu\nu}) \tag{11}$$

and

$$D=-\mathrm{i}/(4\pi^2x^2). \tag{12}$$

The quantity $D^{(0)}_{\alpha\beta\mu\nu}$ is the graviton propagator. The generating functional can now be written as

$$Z[j_{\mu\nu}]=\int \mathrm{D}\phi_{\mu\nu}\,\Delta[\tilde{g}^{\mu\nu}]\exp(\mathrm{i}[I_G^{(0)}+I_G^{(1)}+K\langle j_{\mu\nu}\phi_{\mu\nu}\rangle]). \tag{13}$$

Turning now to the factor $\Delta[\tilde{g}^{\mu\nu}]$, one can express its inverse as follows

$$\Delta^{-1}[\tilde{g}^{\mu\nu}] = \int \mathrm{D}\xi_\lambda\, \mathrm{D}\eta_\nu \exp(\mathrm{i}\langle \eta_\nu \eta_{\mu\nu} \partial^2 \xi_\mu \rangle - \mathrm{i}K\langle \eta_\nu [\phi_{\mu\nu,\lambda\mu} - \phi_{\mu\rho}\eta_{\nu\lambda}\partial_\mu\partial_\rho - \phi_{\mu\rho,\mu}\eta_{\nu\lambda}\partial_\rho + \phi_{\mu\nu,\mu}\partial_\lambda]\xi_\lambda\rangle). \tag{14}$$

The interaction terms between graviton and ghosts can be read off from the above expression. In addition, the ghost propagator is seen to be

$$\langle 0|T[\xi_\mu(x)\eta_\nu(0)]|0\rangle = \eta_{\mu\nu}D(x). \tag{15}$$

Finally, a factor (-1) has to be inserted for each ghost-loop in perturbation series since it is $\Delta[\tilde{g}_{\mu\nu}]$ that occurred in (1) whereas (14) represents $\Delta^{-1}[\tilde{g}_{\mu\nu}]$. The self-energy can now be computed by taking functional derivatives. The five trilinear couplings of the gravitons in (6) give rise to 15 distinct loop-contributions each of which has 18 terms due to identical particles. By contrast, there are only 10 distinct ghost-loop contributions.

The central piece in this self-energy calculation is the Fourier transform of the product of derivatives of the D-functions. Thus, after transition to Euclidean space one must evaluate integrals of the form

$$\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx} D_{,\alpha\beta\ldots\delta}(x) D_{,\mu\nu\ldots\tau}(x) \tag{16}$$

where the number of derivatives on the two D-functions might be partitioned as $(0,2)$, $(1,1)$, $(1,2)$ or $(2,2)$. Unfortunately all these products of derivatives of D functions are very singular so that, as it stands, the above expression is quite meaningless. If one attempts to assign meaning to these divergent integrals by cutting them off at a length l, the results will be proportional to l^{-2} or l^{-4}. Such a cutoff would have to be followed by carefully devised subtractions which in turn would have to be attributed to counter terms in the Lagrangian. There is still the matter of producing a correction to the Green function that satisfies the SWI. Clearly, this is hardly an approach suited to the problem at hand.

To explain our method, we begin with the observation that the products of derivatives of D-functions can be written as follows:

$$D_{,\alpha\beta\ldots\delta}(x) D_{,\mu\nu\ldots\tau}(x) = P^{(n)}_{\alpha\beta\ldots\tau}(\partial)[D(x)]^2. \tag{17}$$

The above result holds for $x \neq 0$ and $P^{(n)}_{\alpha\beta\ldots\tau}(\partial)$ is a tensor differential operator of rank n constructed solely out of ∂_σ, $\delta_{\mu\nu}$ and numerical coefficients. In the case under consideration, $n = 2$, 3 or 4. In reflecting upon the mathematical identity (17) one has to concede that empirical knowledge validates the representation of the propagators and their products as they are used in this identity only through length scales of the order of 10^{-18} m whereas in evaluating (16) we must assume it for all short distances. Here we invoke the crucial physical assumption that the quantity $[D(x)]^2$ which is believed to behave as $(x^2)^{-2}$ at short distances, actually vanishes if x^2 is less than $4l^2$. Accordingly, the replacement

$$[D(x)]^2 \Rightarrow \theta(x^2 - 4l^2)[D(x)]^2 \tag{18}$$

is in order. Consistency demands now that the product of derivatives of the propagators occurring on the left-hand side of (17) be evaluated by making the above replacement. This is our definition of the generic product of derivatives of propagators occurring in (16). Introducing (17) and (18) into (16) and integrating n times by parts we shall obtain

$$\int d^4x\, e^{-ikx}\, D_{,\alpha\beta\ldots\delta}(x) D_{,\mu\nu\ldots\tau}(x)$$

$$\Rightarrow Q^{(n)}_{\alpha\beta\ldots\tau}(k) \int d^4x\, e^{-ikx}[D(x)]^2\theta(x^2-4l^2). \tag{19}$$

The integral on the right-hand side has already been evaluated (Zaidi 1990). The Minkowski space result is approximately

$$J = \int d^4x\, e^{-ikx}[D(x)]^2\theta(x^2-4l^2) \approx -i(16\pi^2)^{-1}\log[l^2(k^2-i\varepsilon)]. \tag{20}$$

The quantity $Q^{(n)}_{\alpha\beta\ldots\tau}(k)$ is a tensor of rank n constructed out of the components of the vector k. All such tensors that are needed in this work are compiled in appendix B. We now have a well-defined expression for all integrals of the form (16) and expect the self-energy to involve J as a factor. As discussed previously (Zaidi 1990), graphs with seagull or tadpole types of structure do not contribute in the framework of the above physical assumption. It remains to perform a perfectly well defined and straightforward computation. Whether the result satisfies the SWI remains to be seen. It is gratifying to see that it does. This is not trivial. For example, we show in the next section that a scalar meson loop insertion violates the SWI unless one uses the improved energy tensor of Coleman and Jackiw (1970). To proceed, the graviton self-energy $\Pi_{\mu\nu\omega\tau}(x-y)$ *is defined by sthe equation*

$$\left(\frac{\delta^2 Z^{(2)}}{\delta J_{\mu\nu}(x)\delta J_{\omega\tau}(y)}\right)$$

$$= \left(\frac{\delta^2 Z^{(0)}}{\delta J_{\mu\nu}(x)\delta J_{\omega\tau}(y)}\right) + \left(\frac{\delta^2 Z^{(0)}}{\delta J_{\mu\nu}(x)\delta J_{\alpha\beta}(u)}\right) i\Pi(u-v)\left(\frac{\delta^2 Z^{(0)}}{\delta J_{\alpha\beta}(v)\delta J_{\omega\tau}(y)}\right). \tag{21}$$

Here, $Z^{(0)}$ and $Z^{(2)}$ are the generating functionals including terms of order K^0 and K^2 respectively and the rest of the notation is standard. In appendix A, we give details and do part of the calculation of the graviton-loop contribution in explicit terms. Here, in order to present the results, we first introduce the notation

$$\Pi^{(\text{graviton})}_{\mu\nu\omega\tau}(k) = \Pi^{(\text{graviton-loop})}_{\mu\nu\omega\tau}(k) + \Pi^{(\text{ghost-loop})}_{\mu\nu\omega\tau}(k).$$

All computed expressions for the various self-energy contributions in this work can be specified in terms of five constants A, B, C, E and F occurring in the following formula:

$$\begin{aligned}\Pi_{\mu\nu\omega\tau}(k) = \frac{K^2}{3840\pi^2}[&Ak_\mu k_\nu k_\omega k_\tau + Bk^2(\eta_{\mu\nu}k_\omega k_\tau + \eta_{\omega\tau}k_\mu k_\nu)\\ &+ Ck^2(\eta_{\mu\omega}k_\nu k_\tau + \eta_{\nu\tau}k_\mu k_\omega + \eta_{\nu\omega}k_\mu k_\tau + \eta_{\mu\tau}j_\nu j_\omega)\\ &+ E(k^2)^2(\eta_{\mu\tau}\eta_{\nu\omega} + \eta_{\nu\tau}\eta_{\mu\omega}) + F(k^2)^2\eta_{\mu\nu}\eta_{\omega\tau}]J^{(0)}.\end{aligned} \tag{22}$$

We state our results by giving the constants. Thus,

$$\Pi^{(\text{graviton-loop})}_{\mu\nu\omega\tau}(k) \Leftrightarrow \{440, 130, -80, 85, -55\} \tag{23}$$

and

$$\Pi^{(\text{ghost-loop})}_{\mu\nu\omega\tau}(k) \Leftrightarrow \{-112, -26, -1, -4, -4\}. \tag{24}$$

The net contribution is accordingly

$$\Pi^{(\text{graviton})}_{\mu\nu\omega\tau}(k)\Leftrightarrow\{328,\ 104,\ -81,\ 81,\ -59\}. \tag{25}$$

Finally, the graviton Green function including the graviton-loop insertion is

$$\tilde{D}^{(2)}_{\mu\nu\omega\tau}(k)=D^{(0)}_{\mu\nu\omega\tau}(k)+\frac{K^2}{3840\pi^2}(k^2)^{-2}T_{\mu\nu\omega\tau}(k)\log[l^2(k^2-\mathrm{i}\varepsilon)] \tag{26}$$

where the tensor $T_{\mu\nu\omega\tau}(k)$ is defined by

$$T_{\mu\nu\omega\tau}(k)=b_{\mu\nu\sigma\rho}S_{\sigma\rho\alpha\beta}(k)b_{\alpha\beta\omega\tau} \tag{27}$$

and the tensor $S_{\sigma\rho\alpha\beta}$ is composed of the square bracket of (22) with the constants given by (25). We can represent $T_{\mu\nu\omega\tau}(k)$ as follows.

$$T_{\mu\nu\omega\tau}(k)\Leftrightarrow\{328,-106,-81,81,46\}. \tag{28}$$

The correction to the Green function due to the graviton loop has different structure from those due to photon, neutrino and scalar meson loops. The last three are proportional to each provided the scalar mesons are coupled through the improved energy tensor. In particular, the coefficient F is positive for the graviton loop correction and negative for the other cases. Consequently, in situations governed by Newton's law of gravitation, the graviton loop contribution to vacuum polarization has a different sign as compared with the other loops mentioned. Admittedly, this is a minute effect. Using (28) one can verify that

$$k_\mu T_{\mu\nu\omega\tau}(k)k_\omega=0. \tag{29}$$

Consequently, the graviton Green function satisfies the SWI:

$$k_\mu\tilde{D}_{\mu\lambda\alpha\beta}(k)k_\alpha=k_\mu D^{(0)}_{\mu\lambda\alpha\beta}(k)k_\alpha=-\tfrac{1}{2}\eta_{\lambda\beta}. \tag{30}$$

At this point one might wonder whether the SWI is also satisfies for values of the constants other than those given in (25). To explore this we consider the tensor $T_{\mu\nu\omega\tau}(k)$ to be given in terms of arbitrary constants A, B, C, E and F and derive the constraints imposed upon them by the requirement that the Green function satisfy the SWI. In this manner, (29) yields

$$\tfrac{1}{4}A-B+E+F=0 \tag{31}$$

and

$$C+E=0. \tag{32}$$

These two conditions are clearly satisfied by the constants in (25).

Our results bear the following relationship to those of Capper *et al* (1973). Their computed graviton-loop and ghost-loop contributions can also be represented by (22). They give explicit formulae for the quantities corresponding to our constants A, B, C, E and F as rational function of ω ($\omega=2$ corresponds to four dimensions). In place of our integral J given by (20), their formulae contain the integral that is basic to the DR method

$$I_1=\int \mathrm{d}^{2\omega}q[q^2(q-p)^2]^{-1}.$$

It is interesting to note that tensors multiplying I_1 in their work if evaluated for $\omega=2$ agree separately with our tensors for the ghost-loop and the graviton-loop. However, they have to expand the counterpart of the entire right-hand side of (27) about $\omega = 2$ to separate the correction to the Green function into a finite part and a pole term. There are also some subtleties involved in continuing the amplitudes to the neighbourhood of $\omega = 2$.

3. Scalar meson loop insertion

The standard formulation of scalar field theory leads to the energy tensor

$$T_{\mu\nu} = \phi_\mu\phi_\nu - \tfrac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial_\rho\phi. \tag{33}$$

It is a source of gravitons in the Einstein theory and gives rise to the self-energy contribution

$$\mathrm{i}\Pi^{(\text{scalar})}_{\alpha\beta\mu\nu}(k) = \frac{K^2}{4}\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx}[D_{,\mu\alpha}D_{,\nu\beta} + D_{,\mu\beta}D_{,\nu\alpha} - \eta_{\mu\nu}D_{,\alpha\sigma}D_{,\beta\sigma} - \eta_{\alpha\beta}D_{,\mu\rho}D_{,\nu\rho} + \tfrac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}D_{,\sigma\rho}D_{,\sigma\rho}]. \tag{34}$$

Using (19) and the quantities $Q^{(4)}_{(\mu\nu)(\omega\tau)}(k)$ given in appendix B, it is straightforward to obtain

$$\Pi^{(\text{scalar})}_{\mu\nu\omega\tau} \Leftrightarrow \{4, -3, -\tfrac{1}{2}, \tfrac{1}{2}, 3\}. \tag{35}$$

Evidently, the scalar-loop insertion does not satisfy (31) so the SWI is violated.

That the scalar field energy tensor leads to additional difficulties in field theory has been the subject of investigation by Coleman and Jackiw (1970) and Callan *et al* (1970). On the other hand, the improved energy tensor of Coleman and Jackiw (1970) was shown by Callan *et al* (1970) to have finite matrix elements in all orders of renormalized perturbation theory. In addition, the currents associated with scale and conformal transformations have much simpler expressions in terms of the improved tensor. However, their proof has been criticized (Symanzik 1970). Furthermore, 't Hooft and Veltman (1974) made a study of all one-loop divergences of pure gravity and of gravitation interacting with scalar fields and concluded that even though the improved energy tensor reduces the divergences of diagrams without internal gravitons, divergences that cannot be absorbed in a field renormalization still remain. Callan *et al* (1970) formulate the theory of gravitation such that the *improved* energy tensor appears as the source of gravitons and show that the theory meets all the experimental tests that have been applied to the general theory of relativity (see also Deser 1970). Thus, it is important to evaluate the self-energy using the improved energy tensor which can be written as

$$\Theta_{\mu\nu} = T_{\mu\nu} - \tfrac{1}{6}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2. \tag{36}$$

Its contribution to self-energy is

$$\mathrm{i}\Pi^{(\text{Coleman-Jackiw})}_{\mu\nu\omega\tau}(k) = \frac{K^2}{72}\int \mathrm{d}^4x\, \mathrm{e}^{-\mathrm{i}kx}[16D_{,\mu\omega}D_{,\nu\tau} - 8\eta_{\omega\tau}D_{,\mu\sigma}D_{,\nu\sigma} - 16D_{,\mu}D_{,\nu\omega\tau} + 4\eta_{\mu\nu}D_{,\sigma}D_{,\omega\tau\sigma} + \eta_{\mu\nu}\eta_{\omega\tau}D_{,\sigma\rho}D_{,\sigma\rho} + 2D_{,\mu\nu}D_{,\omega\tau} + 2DD_{,\mu\nu\omega\tau}]_{\text{sym.}}. \tag{37}$$

The expression inside the square brackets is to be symmetrized with respect to interchange of $(\mu \leftrightarrow \nu)$, $(\omega \leftrightarrow \tau)$ and with respect to $(\mu\omega) \leftrightarrow (\omega\tau)$. Following steps already described, we find

$$
\begin{aligned}
\Pi^{(\text{Coleman-Jackiw})}_{\mu\nu\omega\tau}(k) \\
&= \frac{K^2}{23040\pi^2}[4k_\mu k_\nu k_\omega k_\tau + 2k^2(\eta_{\mu\nu}k_\omega k_\tau + \eta_{\omega\tau}k_\mu k_\nu) \\
&\quad - 3k^2(\eta_{\mu\omega}k_\nu k_\tau + \eta_{\nu\tau}k_\mu k_\omega + \eta_{\nu\omega}k_\mu k_\tau + \eta_{\mu\tau}k_\nu k_\omega) \\
&\quad + 3(k^2)^2(\eta_{\mu\tau}\eta_{\nu\omega} + \eta_{\nu\tau}\eta_{\mu\omega}) - 2(k^2)^2\eta_{\mu\nu}\eta_{\omega\tau}]. \qquad (38)
\end{aligned}
$$

This self-energy is, up to an overall numerical factor, the same as that for the photon or the neutrino loop (Zaidi 1990). It follows that

$$b_{\mu\nu\alpha\beta}\Pi^{(\text{Coleman-Jackiw})}_{\alpha\beta\gamma\delta}(k)b_{\gamma\delta\omega\tau} = \Pi^{(\text{Coleman-Jackiw})}_{\mu\nu\omega\tau}(k) \qquad (39)$$

and

$$k_\mu \Pi^{(\text{Coleman-Jackiw})}_{\mu\nu\omega\tau}(k) = 0. \qquad (40)$$

Furthermore, this loop insertion shares with the photon-loop and the neutrino-loop insertions the important property

$$\Pi^{(\text{Coleman-Jackiw})}_{\mu\nu\sigma\sigma}(k) = 0. \qquad (41)$$

In summary if scalar mesons are coupled to gravitons through the improved energy tensor, the SWI is satisfied and the self-energy as well as the correction to the graviton Green function is traceless. None of these statements is true for the ordinary scalar loop insertion of (35) nor does it satisfy (39).

4. Conclusions

Calculation of the corrections to the graviton Green function is quite a test of the regularization methods used to ascribe meaning to the divergent integrals encountered in field theory calculations. Thus, Halpern (1966) based his work on techniques that were available before the DR method was developed. Later Brown (1973) applied this method to the calculation of the graviton-loop insertion. However the role of the tadpole terms in the context of DR was not well understood at that time. Complete calculations of the graviton, photon and neutrino loop insertions based on the DR method were finally performed respectively by Capper *et al* (1973, 1974) and Capper and Duff (1973). These authors also investigated the tadpole graphs and problems associated with analytic continuation of the amplitudes as functions of ω. The aim of this work was to apply our method to compute the corrections to the graviton Green function. The graviton-loop insertion produces a result that satisfies the SWI identity. No renormalization is performed, nor is it needed. Consequently, no counterterms need be introduced into the Einstein Lagrangian. Scalar-loop insertion is computed along the same lines using both the usual scalar-field tensor and the improved energy tensor. We find that even though we do not have divergences, the ordinary scalar-field tensor is unacceptable because it leads to a violation of the SWI and a self-energy that is not traceless. The improved energy tensor does not have these defects.

This new approach to the problem of divergences has now been applied to most of the self-energies occurring in theories of physical interest. Extensions of this idea to three-point functions and eventually to higher orders is most desirable and planned. Finally, it would be very interesting to create a view of spacetime which justifies our basic assumption.

Appendix A

A part of the calculation which is typical of the manipulations involved in computing the graviton-loop contribution is discussed here in some detail. We choose to consider the graviton loop that has as vertices the first and the third terms of the trilinear couplings given in (6). This contribution to self-energy can be written as

$$\mathrm{i}\Pi^{(13)}_{mnpq}(k)=\left(\frac{K}{2}\right)^2(4)\{\phi_{\gamma\delta}(u)\phi_{\alpha\beta,\gamma}(u)\phi_{\alpha\beta,\delta}(u)\phi_{\omega\tau}(v)\phi_{\omega\sigma,\rho}(v)\phi_{\tau\rho,\sigma}(v)\}_{mnpq}. \tag{A1}$$

The factor of 4 arises because the coefficient of the third trilinear coupling is 2 and because there two such terms, in other words we have written $\Pi^{(13)}$ for $\Pi^{(13)}+\Pi^{(31)}$. The curly bracket stands for the well known process of reducing the right-hand side to the Fourier transform of the products of vacuum expectation values. The result consists of nine terms each of which is composed of a factor arising from the action of zero, one or two derivatives on the external lines carrying the momentum k. The other factor consists of a sum of two terms of the form $(-1)^n b_{\alpha\beta\gamma\delta}b_{\kappa\lambda\mu\nu}\{D_\Xi(u)D_\Omega(u)\}$. Here Ξ and Ω stand for groups of indices representing zero, one or two derivatives and n is the number of derivatives with respect to v acting upon the propagators before translation invariance is used and the result expressed entirely in terms of u. Finally, the curly bracket now stands for just the the Fourier transform. Collecting terms one can write

$$\Pi^{(13)}_{mnpq}=T_1+2T_2+4T_3+2T_4 \tag{A2}$$

where

$$T_1=(b_{\alpha\beta\rho\sigma}b_{\alpha\beta q\rho}\{D_{,m\rho}D_{,n\sigma}\}+b_{\alpha\beta\rho\omega}b_{\alpha\beta s\omega}\{D_{,m\sigma}D_{,n\rho}\}) \tag{A3}$$

$$T_2=-\mathrm{i}k_\sigma(b_{\alpha\beta\rho\omega}b_{\alpha\beta\sigma\omega}\{D_{,m}D_{,nq}\}+b_{\alpha\beta\rho\omega}b_{\alpha\beta\sigma\omega}\{D_{,n}D_{,mq}\}) \tag{A4}$$

$$T_3=-k_\gamma k_\sigma(b_{\mu\mu\rho\omega}b_{\gamma\delta\sigma\omega}\{D_{,\delta}D_{,q}\}+b_{\beta\delta\rho\omega}b_{\mu\nu\sigma\omega}\{DD_{,\delta q}\}) \tag{A5}$$

and

$$T_4=-\mathrm{i}k_\gamma(b_{\mu\nu\rho\sigma}b_{\gamma\delta q\rho}\{D_{,\sigma}D_{,\delta\rho}\}+b_{\gamma\delta\rho\sigma}b_{\mu\nu q\rho}\{D_{,\rho}D_{,\delta\sigma}\}). \tag{A6}$$

Using (19) and the quantities $Q^{(n)}_{\alpha\beta\ldots\tau}(k)$ given in appendix B one can complete the calculation which now involves only algebraic manipulation. The result is of the form of (22) of the text so it can be specified by giving the coefficients A through F:

$$\Pi^{(13)}_{\mu\nu\omega\tau}\Leftrightarrow\{128,-116,-21,21,11\}. \tag{A13}$$

The other 14 distinct contributions to the graviton loop are evaluated by following the same pattern. The 10 distinct ghost-loop contributions are simpler.

Appendix B

The list of $Q^{(n)}_{\alpha\beta\ldots\tau}(k)$ needed in this work are given in the following. The method used to construct them is based on the observation that the indices of the tensor differential operator $P^{(n)}_{\alpha\beta\ldots\tau}(\partial)$ of (17), must have the same permutation symmetry as the indices on the left-hand side of that equation. Then, it is only a matter of making an ansatz and determining the coefficients.

$$Q^{(2)}_{(\mu)(\nu)}(k) = -\tfrac{1}{12}(2k_\mu k_\nu + \eta_{\mu\nu}k^2) \tag{B1}$$

$$Q^{(2)}_{()(\mu\nu)}(k) = -\tfrac{1}{12}(4k_\mu k_\nu - \eta_{\mu\nu}k^2) \tag{B2}$$

$$Q^{(3)}_{(\lambda)(\mu\nu)}(k) = \frac{-\mathrm{i}}{24}[2k_\lambda k_\mu k_\nu + k^2(\eta_{\lambda\mu}k_\nu + \eta_{\lambda\nu}k_\mu - \eta_{\mu\nu}k_\lambda)] \tag{B3}$$

$$\begin{aligned} Q^{(4)}_{(\mu\nu)(\omega\tau)}(k) = \tfrac{1}{240}[&8k_\mu k_\nu k_\omega k_\tau - 6k^2(\eta_{\mu\nu}k_\omega k_\tau + \eta_{\omega\tau}k_\mu k_\nu) \\ &+ 4k^2(\eta_{\mu\omega}k_\nu k_\tau + \eta_{\nu\tau}k_\mu k_\omega + \eta_{\nu\omega}k_\mu k_\tau + \eta_{\mu\tau}k_\nu k_\omega) \\ &+ (k^2)^2(\eta_{\mu\tau}\eta_{\nu\omega} + \eta_{\nu\omega}) + (k^2)^2\eta_{\mu\nu}\eta_{\omega\tau}]. \end{aligned} \tag{B4}$$

$$\begin{aligned} Q^{(4)}_{()(\mu\nu\omega\tau)}(k) = \tfrac{1}{240}[&48k_\mu k_\omega k_\omega k_\tau - 6k^2(\eta_{\mu\nu}k_\omega k_\tau + \eta_{\omega\tau}k_\mu k_\omega) \\ &- 6k^2(\eta_{\mu\omega}k_\nu k_\tau + \eta_{\nu\tau}k_\mu k_\omega + \eta_{\nu\omega}k_\mu k_\tau + \eta_{\mu\tau}k_\nu k_\omega) \\ &+ (k^2)^2(\eta_{\mu\tau}\eta_{\nu\omega} + \eta_{\nu\tau}\eta_{\mu\omega}) + (k^2)^2\eta_{\mu\nu}\eta_{\omega\tau}]. \end{aligned} \tag{B5}$$

$$\begin{aligned} Q^{(4)}_{(\kappa)(\lambda\mu\nu)}(k) = \tfrac{1}{240}[&12k_\kappa k_\lambda k_\mu k_\nu - 4k^2(\eta_{\mu\nu}k_\kappa k_\lambda + \eta_{\lambda\nu}k_\kappa k_\mu + \eta_{\lambda\mu}k_\kappa k_\nu) \\ &+ 6k^2(\eta_{\kappa\nu}k_\lambda k_\mu + \eta_{\kappa\mu}k_\lambda k_\nu + \eta_{\kappa\lambda}k_\mu k_\nu) \\ &- (k^2)^2(\eta_{\mu\lambda}\eta_{\nu\kappa} + \eta_{\nu\lambda}\eta_{\mu\kappa} + \eta_{\mu\nu}\eta_{\kappa\lambda})]. \end{aligned} \tag{B6}$$

The indices in the two parentheses correspond to the derivatives acting on the two D-functions. Empty parentheses arise if a D-function is not differentiated. Equations (B5) and (B6) are needed in connection with the improved energy tensor.

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